The Studies of the Stability of FDTD with Mur's Absorbing Boundary Condition of Second Order in 3-D Scattering Problems

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***Abstract*—In this letter, the numerical implementation of Mur's absorbing boundary condition (Mur's ABC) of second order in finite-difference time-domain (FDTD) is analyzed and studied in three-dimensional (3-D) scattering problems. Two new stable upwind finite-difference approximations of Mur's ABC are presented and the later-time instability of FDTD with second-order Mur's ABC in 3-D scattering problems is overcomed. Numerical results clearly exhibit the availability of the two upwind finite-difference approximations of Mur's ABC.**

I. Introduction

After the finite-difference time-domain (FDTD) method was first proposed by Yee in 1966 [1] to solve electromagnetic scattering problems, many types of absorbing boundary conditions have been presented to overcome the difficulty arising from the fact that scattering problems are usually open problems, i.e., the domain in which the field has to be computed is unbounded [2]–[5]. One of these absorbing boundary conditions was proposed by Mur in 1981 and called Mur's ABC [3]. Since then, the central finite-difference approximation of Mur's ABC has been widely used in the FDTD technique, due to its easy programming and high accuracy. However, the central finite-difference approximation of second-order Mur's ABC combined with the FDTD model to solve three-dimensional (3-D) scattering problems is instable on later time; this limits its applicability. When the incident wave is the sinusoidal wave in the FDTD model and assumed to be turned on at $t = 0$ instant, time stepping is continued until sinusoidal steady-state field values are observed at all grids within the space computed domain. This time interval is dependent on the electrical dimensions of the scatterer. The larger the scatterer is in electrical dimensions, the longer the time interval needed. So, the later-time stability of absorbing boundary conditions combined with FDTD method is very important and interesting problem, especially for larger scatterers in electrical dimension. In Section II, two new stable finite-difference approximations of Mur's absorbing boundary condition (Mur's ABC) are presented. Numerical examples are given in the Section III. At the last part of this paper, we get the valuable conclusions.

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II. Different Types of Finite-Difference Approximations of Mur's ABC

For 3-D FDTD model, the Yee's space cell is used to discrete the computed domain in rectangular Cartesian coordinates. Following Yee's notation, we denote a space grid point as $(i, j, k) = (i\Delta x, j\Delta y, k\Delta z)$ and any function of space and time as $w^n(i,j,k) = w(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$, where $\Delta X, \Delta y, \Delta z$ is the space increment in $\hat{x}, \hat{y}, \hat{z}$ direction, respectively, and Δt is time increment. We will assume that the space grides are located in the region $x \geq 0$. The Mur's ABC of second order at boundary $x = 0$ plane can be written as

$$\left\{\frac{\partial^2}{c\partial x\partial t} - \frac{\partial^2}{c^2\partial t^2} + \frac{1}{2}\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial Z^2}\right)\right\}W|_{x=0} = 0. \quad (1)$$

Let $\Delta x = \Delta y = \Delta z = \Delta$, and the different discretized forms of the Mur's ABC (1) are derived as follows.

1) By means of central-difference expressions for the space and time derivatives in (1) at $x = \Delta/2$ plane, the Mur's ABC reads

$$\begin{aligned} &W^{n+1}(0,j,k) \\ &\quad = -W^{n-1}(1,j,k) + \frac{c\Delta t = \Delta}{c\Delta t + \Delta}\{W^{n+1}(1,j,k) \\ &\qquad + W^{n-1}(0,j,k)\} + \frac{2\Delta}{c\Delta t + \Delta}\{W^n(1,j,k) \\ &\qquad + W^n(0,j,k)\} + \frac{(c\Delta t)^2}{2(c\Delta t + \Delta)\cdot\Delta}\{W^n(0,j+1,k) \\ &\qquad + W^n(0,j-1,k) + W^n(0,j,k+1) \\ &\qquad + W^n(0,j,k-1) - 4W^n(0,j,k)W^n(1,j+1,k) \\ &\qquad + W^n(1,j-1,k) + W^n(1,j,k+1) \\ &\qquad + W^n(1,j,k-1) - 4W^n(1,j,k)\}. \quad (2) \end{aligned}$$

This equation was first derived by Mur in 1981.

2) By means of central-difference approximation for the space and the time derivative in the first item and the second item in (1) at $x = \Delta/2$ plane and the central-difference approximation for the space derivative in the third item at $x = 0$ plane, the Mur's ABC is discretized as

$$\begin{aligned} &W^{n+1}(0,j,k) \\ &\quad = -W^{n-1}(1,j,k) + \frac{c\Delta t - \Delta}{c\Delta t + \Delta}\{W^{n+1}(1,j,k) \\ &\qquad + W^{n-1}(0,j,k)\} + \frac{2\Delta}{c\Delta t + \Delta}\{W^n(1,j,k) \end{aligned}$$

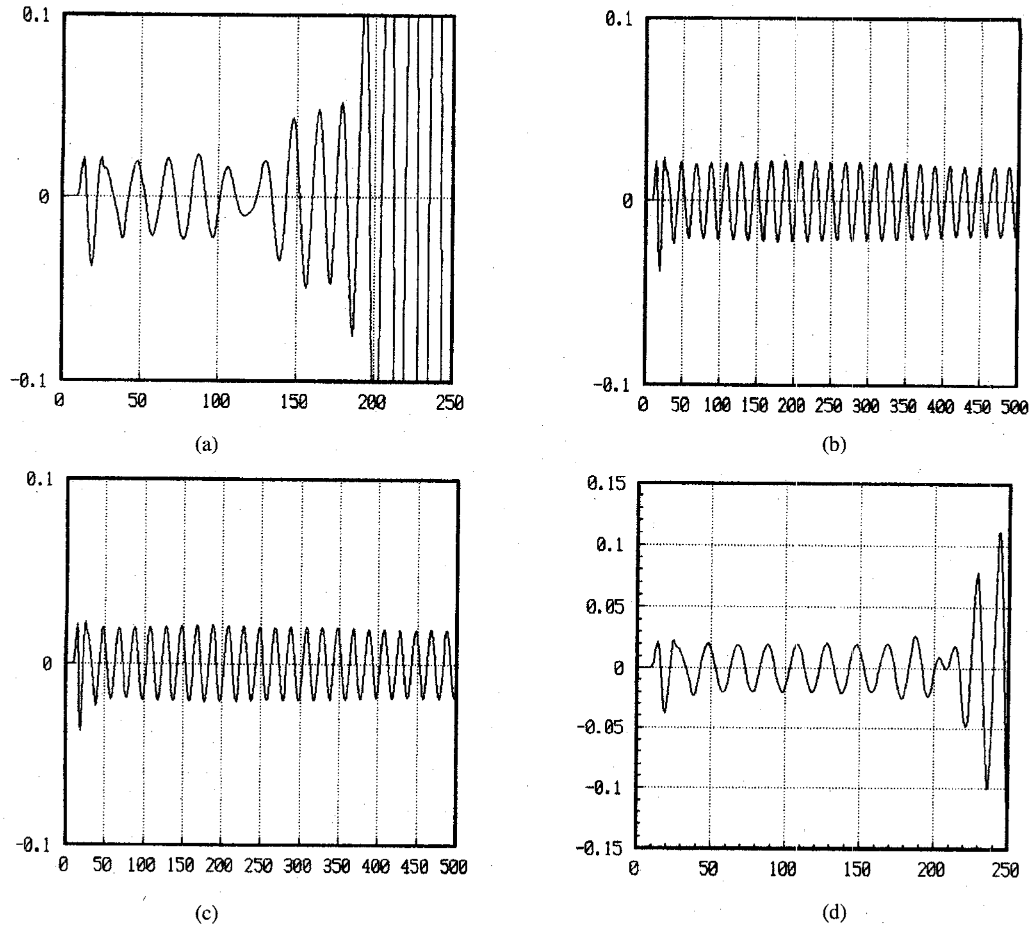

Fig. 1. The time waveform of E_x at point A computed by FDTD with different types of finite-difference approximation of second-order Mur's ABC when the source is a sinusoidal electric dipole.

$$\begin{aligned}&+ W^n(0,j,k)\} + \frac{(c\Delta t)^2}{(c\Delta t + \Delta)\cdot\Delta}\{W^n(0,j+1,k)\\&+ W^n(0,j-1,k) + W^n(0,j,k+1)\\&+ W^n(0,j,k-1) - 4W^n(0,j,k)\}.\end{aligned} \tag{3}$$

3) By means of upwind-difference approximation for first item and second item of (1) and central-difference approximation for third item of (1) at $x = 0$ plane, the Mur's ABC is discretized as

$$\begin{aligned}&W^{n+1}(0,j,k)\\&\quad = 2W^n(0,j,k) - W^{n-1}(0,j,k)\\&\quad - \frac{c\Delta t}{\Delta}\{W^n(0,j,k) - W^n(1,j,k)\}\\&\quad + \frac{c\Delta t}{\Delta}\{W^{n-1}(0,j,k) - W^{n-1}(1,j,k)\}\\&\quad + \frac{(c\Delta t)^2}{2\Delta^2}\{W^n(0,j+1,k)\\&\quad + W^n(0,j-1,k) - W^n(0,j,k+1)\\&\quad + W^n(0,j,k-1) - 4W^n(0,j,k)\}.\end{aligned} \tag{4}$$

Note that the equation have a local truncation error of the second order in all increments due to using the central-difference approximations for space and time derivatives, but (3) and (4) have a local truncation error of order $(\Delta + \Delta t^2)$ and order $(\Delta + \Delta t)$, respectively, because of the upwind approximations for space and time derivatives.

III. Numerical Results

Without loss of generality, we shall assume that in free space the exciting source located at the center grid $O(Ic, Jc, Kc)$ in computed domain is a sinusoidal electric dipole in $\hat{x}$ direction, i.e.

$$\vec{E}^i = E_x^i\hat{x} = \sin(2\pi c/\lambda \cdot t)\cdot\hat{x} \tag{5}$$

where c denotes the speed of light in vacuo and λ is the wavelength of the source. We shall investigate the stability of (2)–(4) when combined with the FDTD method. In the FDTD numerical model let $\Delta = 0.1\lambda$ (the grid size) and let $L = \lambda$, (the distance between outer boundary and dipole source in all three axis directions, i.e., 10 grids). The numerical Mur's ABC (2)–(4) at the truncation boundary is used, respectively, the electric component E_x at grid point A $(Ic, Jc+5, Kc+3)$ is observed and shown in Fig. 1(a)–(c). (The other field components at any grid points in computed domain can be observed also.) Let $L = 1.5\lambda$, where the numerical boundary condition (2) is used at outer boundary and the electric component E_x at A is shown in Fig. 1(d). It is clear from this figure that the approximation (2) of Mur's ABC introduces later-time numerical instability in the problem, and that the approximation (3) or (4) of Mur's ABC removes this quickly growing instability. Fig. 1(a) and (d) shows that the numerical instability can be minimized, but not removed, by increasing the separation between the source and the boundary when the FDTD method is combined with central-difference approximation (2) of Mur's ABC. This numerical instability is not due to multiple reflections from the outer boundary but the difference approximation forms.

When the dipole source is excited by a Gaussian pulse current, that is

$$\vec{E}^i = E_x^i \hat{x} = e^{-\{(t-t_0)/t_1\}^2} \cdot \hat{x} \tag{6}$$

where $t_0 = 50\Delta t, t_1 = 12\Delta t$. The electric component E_x at grid point A computed by FDTD method with Mur's ABC (2),(3) and (4) respectively is shown in Fig. 2. The grid size is 0.1 m and time step Δt equals 1.5 ns. The computed domain is $20 \times 20 \times 20$. We find that the upwind-difference approximation of Mur's ABC has better stability than the central-difference approximation.

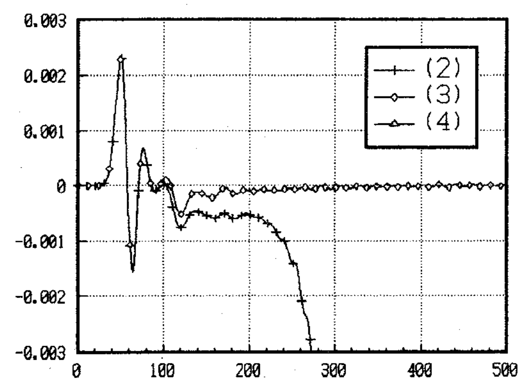

Fig. 2. The time waveform of E_x at point A computed by FDTD with different types of finite-difference approximations of second-order Mur's ABC when the source is a Gaussian pulse electric dipole.

IV. Conclusion

When FDTD technique is used to solve 3-D open electromagnetic problems (scattering or radiation problems), the central-difference approximation of second-order Mur's ABC has the disadvantage of later-time instability. For a larger scatterer in electric dimension, the instability is a severe problem and the central-difference approximation of Mur's ABC cannot be used when the later-time solution is desired. The numerical results illustrated that the upwind-difference approximations of second-order Mur's ABC, although not as accurate as the central-difference approximation, have good stability when combined with the FDTD method.

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